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Approximation of Dense-$n/2$-Subgraph and the Complement of Min-Bisection

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Abstract. We consider the DENSE-$n/2$-SUBGRAPH problem, i.e., determine a block of half number nodes from a weighted graph such that the sum of the edge weights, within the subgraph induced by the block, is maximized. We prove that a strengthened semidefinite relaxation with a mixed rounding technique yields a 0.586 approximations of the problem. The previous best-known results for approximating this problem are 0.25 using a simple coin-toss randomization, 0.48 using a semidefinite relaxation, 0.5 using a linear programming relaxation or another semidefinite relaxation. In fact, an un-strengthened SDP relaxation provably yields no more than 0.5 approximation. We also consider the complement of the graph MIN-BISECTION problem, i.e., partitioning the nodes into two blocks of equal cardinality so as to maximize the weights of non-crossing edges. We present a 0.602 approximation of the complement of MIN-BISECTION.

Key words: Min-bisection; Dense-k-subgraph; Polynomial approximation algorithm; Semidefinite programming

1. Introduction

Given an undirected graph $G = (V, E)$ and non-negative weights $w_{ij} = w_{ji}$ on the edges $(i, j) \in E$, the b-balanced min-cut problem is that of finding a subset of nodes or vertices $S \subset V$ to minimize

$$\frac{1}{2} \sum_{(i,j)\in E, i\in S, j\in V\setminus S} w_{ij}$$

such that $bn \leqslant |S| \leqslant (1-b)n$, where $n = |V|$ and $b \leqslant \frac{1}{2}$. The special case in which $b = \frac{1}{2}$ is sometimes referred to as the graph MIN-BISECTION problem (Shmoys, 1996). The problem can be formulated as follows:

$$\text{(GBP)} \qquad \begin{aligned} w_* := \text{Minimize} \quad & \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_i x_j) \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = 0 \quad \text{or} \quad e^T x = 0 \\ & x_j^2 = 1, \; j = 1, \dots, n\,, \end{aligned}$$

where $e \in R^n$ is the column vector of all ones, superscript T is the transpose operator.

(GBP) is a fundamental problem in the area of combinatorial optimization. It arises from many network applications such as circuit partitioning (Choi and Ye, 1999). In particular, it is used as a subroutine in designing approximation algorithms for a class of combinatorial optimization problems, via a divide-and-conquer approach; see Leighton and Rao (1999) and Shmoys (1996).

Leighton and Rao (1988) showed that there exists a polynomial algorithm for finding a b-balanced cut with $b = \frac{1}{3}$ of value $O(\log n) \cdot w_*$, where w_* is the minimal value of the (GBP). Even et al. (1997) get the similar result by using an alternative approach. Note that these are not true approximation algorithms for (GBP) (Williamson, 1998), since $b \neq \frac{1}{2}$. Very recently, in a major breakthrough, a polylogarithmic approximation algorithm for (GBP) was obtained by Feige and Krauthgamer (2000). For the planar graph, there exists a 2-approximation algorithm due to Garg et al. (1994).

Many heuristic methods have been proposed in the literature. In a recent note, Choi and Ye (1999) carried out some computational study of using a semidefinite programming (SDP) relaxation to approximate the Circuit Bisection problem. They essentially modeled the Circuit Bisection problem as a (GBP), and then used a SDP relaxation based method to solve the problem. Their computational results showed that the SDP approach worked better than other known heuristics, especially when circuits are large.

The SDP relaxation has been successfully applied to various graph optimization problems. Goemans and Williamson (1995) presented an 0.878-approximation algorithm for the MAX-CUT problem. Feige and Goemans (1995) refined this approach by adding additional constraints to the semidefinite program and gave an improved 0.859-approximation algorithm for the MAX-DICUT problem. Frieze and Jerrum (1997) obtained polynomial approximation algorithms for MAX-K-CUT and MAX-BISECTION, and both of them have a better performance guarantee than earlier results. Nesterov (1998), Zwick (1999), and Ye (1999b) extended the SDP method to solving more general problems. Using more complicated rounding techniques and analysis, Ye (1999a) made an improvement in the performance guarantee from 0.651 of Frieze and Jerrum (1997) to 0.699 for MAX-BISECTION. (See Mahajan and Ramesh (1999) for some de-randomization techniques for SDP.)

Using the same techniques, we now present a 0.602 approximation for the complement of (GBP), that is,

$$\begin{array}{llll} & w^* := \text{Minimize} & \frac{1}{2}\sum_{i<j} w_{ij}(1 + x_i x_j) & \\ \text{(CGBP)} & \text{subject to} & \sum_{j=1}^{n} x_j = 0 \quad \text{or} \quad e^T x = 0 & \\ & & x_j^2 = 1,\ j = 1, \ldots, n\,. & \end{array}$$

Note that for the same graph, the sum of w_* and w^* is the sum of the total edge weights of the graph:

$$w^* + w_* = \sum_{i<j} w_{ij} .$$

A closely related problem to (CGBP) is the so-called DENSE-k-SUBGRAPH problem (DSP), i.e., determine a subset $S \subset V$ of k nodes such that the total weight of the subgraph induced by S is maximized, see Srivastav and Wolf (1998). This problem can be formulated as:

$$\text{(DSP)} \qquad \begin{array}{rl} w^* := \text{Minimize} & \frac{1}{4}\sum_{i<j} w_{ij}(1 + x_i + x_j + x_i x_j) \\ \text{subject to} & \sum_{j=1}^{n} x_j = 2k - n \quad \text{or} \quad e^T x = 2k - n \\ & x_j^2 = 1,\ j = 1, \ldots, n , \end{array}$$

In this paper, we consider the case of $k = n/2$.

The previous best-known results for this problem are 0.25 using a simple coin-toss randomization technique, 0.5 using a linear programming relaxation (Goemans, 1996) or a SDP relaxation (Feige and Seltser, 1997), and 0.48 using a different SDP relaxation (Srivastav and Wolf, 1998). In fact, Srivastav and Wolf have proved that the SDP relaxation used in their analysis yields at most 0.5 approximation of DENSE-$n/2$-SUBGRAPH. Crossing the 0.5 barrier, we present in this paper a 0.586 strengthened SDP-based approximation algorithm, following a 0.519 approximation algorithm, for the DENSE-$n/2$-SUBGRAPH problem.

After the announcement of our results, we have learnt of the 0.517-approximation algorithm for DENSE-$n/2$-SUBGRAPH problem and 0.547-approximation algorithm for the complement of (GBP), which were independently obtained by Feige and Langberg (1999).

2. SDP relaxation and approximation of (CGBP)

Our semidefinite programming relaxation of (CGBP) is:

$$\text{(SDP)} \qquad \begin{array}{rl} w_{SD} := \text{Minimize} & \frac{1}{2}\sum_{i<j} w_{ij}(1 + X_{ij}) \\ \text{subject to} & \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij} = 0 \quad (\text{or} \quad ee^T \bullet X = 0) , \\ & X_{jj} = 1,\ j = 1, \ldots, n,\ X \succcurlyeq 0 . \end{array} \qquad (1)$$

Here, $e \in R^n$ is the vector of all ones and the unknown $X \in R^{n \times n}$ is a symmetric matrix. Furthermore, $\bullet$ is the matrix inner product $Q \bullet X = \text{trace}(Q^T X)$, and $X \succcurlyeq Z$ means that $X - Z$ is positive semidefinite.

Obviously, (SDP) is a relaxation of (CGBP), since for any feasible solution x of (CGBP), $X = xx^T$ is feasible for (SDP); so that $w_{SD} \geqslant w^*$. Let $\bar{X}$ be an optimal solution of (SDP).

The following approximation method is similar to the one in Frieze and Jerrum (1997), Srivastav and Wolf (1998), and Ye (1999a), also see Nesterov (1998) and Zwick (1999).

1. SDP solving: Solve (SDP) to obtain the semidefinite matrix $\bar{X}$.

Repeat the following two steps for $[\epsilon^{-1} \log \epsilon^{-1}]$ times, where $\epsilon > 0$ is a positive small constant as stated in Frieze and Jerrum (1997), and output the best bisection.

2. Randomized rounding: Generate a vector u from a multivariate normal distribution with 0 mean and the covariance matrix $\theta^* \bar{X} + (1 - \theta^*)I$, where I is the identity matrix, and $0 < \theta^* \leqslant 1$ will be specified later in the next section. That is, generate

$$u \in N(0, \theta^* \bar{X} + (1 - \theta^*)I),$$

then assign

$$\hat{x} = \text{sign}(u),$$

i.e.,

$$\hat{x}_i = \begin{cases} 1 & \text{if } u_i \geqslant 0 \\ -1 & \text{if } u_i < 0 . \end{cases}$$

Select the block $S = \{i : \hat{x}_i = 1\}$ or $S = \{i : \hat{x}_i = -1\}$ such that $|S| \geqslant n/2$. Let $\tilde{S} = S$.

3. Node swapping: For each $i \in \tilde{S}$, let $\zeta(i) = \Sigma_{j \in \tilde{S}} w_{ij}$ and $\tilde{S} := \{i_1, i_2, \ldots, i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geqslant \zeta(i_2) \geqslant \cdots \geqslant \zeta(i_{|\tilde{S}|})$. Then, remove the node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, \ldots, i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = n/2$. □

For any $U \subset V$, denote the total weights within the subgraph induced by U as $w(U)$, that is,

$$w(U) := \sum_{i<j,\, i \in U,\, j \in U} w_{ij},$$

define

$$w(U, V \backslash U) = w(U) + w(V \backslash U).$$

Then, the following lemma holds and is due to Srivastav and Wolf (1998):

LEMMA 1. *The total weights of non-crossing edges*

$$w(\tilde{S}, V \backslash \tilde{S}) \geqslant \frac{\frac{n}{2}\left(\frac{n}{2} - 1\right)}{|S|(|S| - 1)} \cdot w(S, V \backslash S). \tag{2}$$

Proof. The edge disappears in the swapping procedure if and only if one of its endpoints is removed. It follows that

$$\sum_{i\in\tilde{S}} w(\tilde{S}\backslash\{i\}) = (|\tilde{S}|-2)w(\tilde{S}) ,$$

since each edge is counted $|\tilde{S}|-2$ times. Suppose v is the node that is removed in the swapping procedure, then

$$\sum_{j\in\tilde{S}} w_{vj} \leqslant \sum_{j\in\tilde{S}} w_{ij} \quad \text{for any other } i\in\tilde{S} .$$

Thus,

$$w(\tilde{S}\backslash\{v\}) \geqslant \frac{1}{|\tilde{S}|}\sum_{j\in\tilde{S}} w(\tilde{S}\backslash\{i\}) = \frac{|\tilde{S}|-2}{|\tilde{S}|}\cdot w(\tilde{S}) .$$

By induction we get that the final $\tilde{S}$ satisfies

$$w(\tilde{S}) \geqslant \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{|S|(|S|-1)}\cdot w(S) .$$

Therefore,

$$\begin{aligned} w(\tilde{S}, V\backslash\tilde{S}) = w(\tilde{S}) + w(V\backslash\tilde{S}) &\geqslant \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{|S|(|S|-1)}\cdot w(S) + w(V\backslash S) \\ &\geqslant \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{|S|(|S|-1)}\cdot w(S, V\backslash S) . \end{aligned}$$

□

3. Analysis of the approximation of (CGBP)

In order to analyze the quality of bisection $\tilde{S}$, we define two random variables similar to those in Frieze and Jerrum (1997) and Ye (1999a):

$$w := w(S, V\backslash S) = \frac{1}{2}\sum_{i<j} w_{ij}(1+\hat{x}_i\hat{x}_j) = \frac{1}{4}\sum_{i\neq j} w_{ij}(1+\hat{x}_i\hat{x}_j)$$

and

$$m := |S|(n-|S|) = \frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4} = \frac{1}{4}\sum_{i,j}(1-\hat{x}_i\hat{x}_j) .$$

LEMMA 2. *Our approximation method yields S satisfying the following two inequalities:*

$$E[w] \geqslant \alpha\cdot w_{SD} \geqslant \alpha\cdot w^* ,$$

$$E[m] \geqslant \beta\cdot\frac{n^2}{4} .$$

where $\alpha := \alpha(\theta^*)$, $\beta := \beta(\theta^*)$, *and*

$$\alpha(\theta) = \min_{-1<y\leqslant 1} \frac{1+\frac{2}{\pi}\arcsin(\theta y)}{1+y}, \tag{3}$$

$$\beta(\theta) = 1-\frac{2}{\pi}\arcsin(\theta) - \frac{1-\frac{2}{\pi}\arcsin(\theta)}{n} + \min_{-1\leqslant y<1} \frac{2}{\pi}\frac{\arcsin(\theta)-\arcsin(\theta y)}{1-y}. \tag{4}$$

Proof. From Lemma 2.2 of Goemans and Williamson (1995), and Proposition 2 of Bertsimas and Ye (1998), we have

$$\mathrm{E}[\hat{x}_i\hat{x}_j] = \frac{2}{\pi}\arcsin(\theta^*\bar{X}_{ij}), \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{5}$$

Using the same argument in Ye (1999a) and by the definition of α from (3), we get

$$1+\frac{2}{\pi}\arcsin(\theta^*\bar{X}_{ij}) \geqslant \alpha(1+\bar{X}_{ij}).$$

Thus

$$\begin{aligned}\mathrm{E}[w] &= \frac{1}{4}\sum_{i\neq j} w_{ij}\left(1+\frac{2}{\pi}\arcsin(\theta^*\bar{X}_{ij})\right) \geqslant \frac{1}{4}\sum_{i\neq j} w_{ij}\cdot\alpha(1+\bar{X}_{ij}) \\ &= \alpha\cdot w_{SD}.\end{aligned} \tag{6}$$

Let

$$\alpha' = \min_{-1\leqslant y<1} \frac{2}{\pi}\cdot\frac{\arcsin(\theta^*)-\arcsin(\theta^* y)}{1-y}$$

and noting that

$$\sum_{i\neq j}\bar{X}_{ij} = -n,$$

we derive

$$\begin{aligned} n^2-\mathrm{E}[(e^T\hat{x})^2] &= \sum_{i\neq j}\left(1+\frac{2}{\pi}\arcsin(\theta^*\bar{X}_{ij})\right) \\ &\geqslant \sum_{i\neq j}\left(1-\frac{2}{\pi}\arcsin(\theta^*)+\alpha'(1-\bar{X}_{ij})\right) \\ &= (n^2-n)\left(1-\frac{2}{\pi}\arcsin(\theta^*)\right)+(n^2-n)\alpha'+n\alpha' \\ &= \left(1-\frac{2}{\pi}\arcsin(\theta^*)+\alpha'\right)\cdot n^2 - \left(1-\frac{2}{\pi}\arcsin(\theta^*)\right)\cdot n \\ &= \left(1-\frac{2}{\pi}\arcsin(\theta^*)+\alpha' - \frac{1-\frac{2}{\pi}\arcsin(\theta^*)}{n}\right)\cdot n^2 \\ &= \beta\cdot n^2. \end{aligned}$$

Thus, the second desired result in the lemma is obtained by the definition of m. □

Consider a new random variable

$$z(\gamma) := \frac{w}{w^*} + \gamma \frac{4m}{n^2}, \tag{7}$$

where

$$\gamma = \frac{\alpha}{\beta}\left(\frac{1}{\sqrt{1-\beta}} - 1\right). \tag{8}$$

By Lemma 2, we have

$$\mathrm{E}[z(\gamma)] \geqslant \alpha + \gamma\beta \quad \text{and} \quad z(\gamma) \leqslant 1 + \gamma .$$

Let θ^* and R^* be the optimizer and optimal value, respectively, to the following:

$$\max_{0 \leqslant \theta \leqslant 1} R(\theta) := \frac{\alpha(\theta)}{\beta(\theta)^2}(1 - \sqrt{1-\beta(\theta)})^2 .$$

Then we have:

LEMMA 3. *If the random variable $z(\gamma)$ fulfills its expectation, i.e., $z(\gamma) \geqslant \alpha + \gamma\beta$, then*

$$w(\tilde{S}, V \backslash \tilde{S}) \geqslant \frac{n-2}{n-1} \cdot R^* \cdot w^* .$$

Proof. Suppose

$$w(S, V\backslash S) = \lambda w^* \quad \text{and} \quad |S| = \delta n \geqslant n/2 ,$$

which from (7) and $z(\gamma) \geqslant \alpha + \gamma\beta$ implies that

$$\lambda \geqslant \alpha + \gamma\beta - 4\gamma\delta(1-\delta) .$$

Applying (2), $1/2 \leqslant \delta \leqslant 1$, and (8), we see that

$$\begin{aligned} w(\tilde{S}, V\backslash\tilde{S}) &\geqslant \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{\delta n(\delta n - 1)} w(S, V\backslash S) \\ &\geqslant \frac{n-2}{n - \frac{1}{\delta}} \cdot \frac{1}{4\delta^2} \cdot w(S, V\backslash S) \\ &\geqslant \frac{n-2}{n-1} \cdot \frac{1}{4\delta^2} \cdot w(S, V\backslash S) \\ &= \frac{n-2}{n-1} \cdot \frac{\lambda}{4\delta^2} \cdot w^* \end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{n-2}{n-1}\cdot\frac{\alpha+\gamma\beta-4\gamma\delta(1-\delta)}{4\delta^2}\cdot w^* \\
&\geqslant \frac{n-2}{n-1}\cdot\frac{\alpha\gamma+(\beta-1)\gamma^2}{\alpha+\gamma\beta}\cdot w^* \\
&= \frac{n-2}{n-1}\cdot\frac{\alpha}{\beta^2}(1-\sqrt{1-\beta})^2\cdot w^* \\
&= \frac{n-2}{n-1}\cdot R^*\cdot w^* .
\end{aligned}$$

The last inequality follows from simple calculus that, if we let

$$\phi(\delta)=\frac{\alpha+\gamma\beta-4\gamma\delta(1-\delta)}{4\delta^2},$$

the only root for $\phi'(\delta)=0$ in $(0,\infty)$ is

$$\delta_0=\frac{\alpha+\beta\gamma}{2\gamma}$$

and

$$\phi''(\delta_0)=\gamma\delta_0^{-3}>0 . \qquad \square$$

Now we are ready to give the first main result of this paper.

THEOREM 1. *The worst case performance ratio of the approximation method for the complement of MIN-BISECTION is at least* 0.602 *as* n *sufficiently large*.

Proof. We verified that for $\theta=0.78$, $\alpha(\theta)\geqslant 0.7848$ and $\beta(\theta)\geqslant 0.9800$ for n sufficiently large, therefore,

$$R^*\geqslant R(\theta)=\frac{\alpha(\theta)}{\beta(\theta)^2}(1-\sqrt{1-\beta(\theta)})^2\geqslant 0.6024 ;$$

Thus, for n sufficiently large, we must have

$$\frac{n-2}{n-1}\cdot R^*\geqslant 0.602 .$$

Following the rest of proof of Frieze and Jerrum (1997), we have proved the theorem. $\square$

4. SDP relaxation and approximation of (DSP)

Our SDP relaxation of (DSP) is

$$
\begin{aligned}
& w_{SD} := \text{Maximize} && \frac{1}{4}\sum_{0<i<j} w_{ij}(1+X_{0i}+X_{0j}+X_{ij}) \\
\text{(DSDP)} \quad & \text{subject to} && \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}=0 \quad (\text{or } (0;e)(0;e)^T \bullet X = 0)\,, \\
& && X_{jj}=1,\ j=0,\dots,n,\ X \succeq 0\,.
\end{aligned} \tag{9}
$$

Here, the unknown $X \in R^{(n+1)\times(n+1)}$, row and column indexed from 0 to n.

The rounding algorithm for (DSP) is similar to Srivastav and Wolf (1998):

1. SDP solving: Solve the problem (DSDP) to obtain a semidefinite matrix $\bar{X}$.

Repeat the following two steps for $[\epsilon^{-1}\log\epsilon^{-1}]$ times, where $\epsilon > 0$ is small constant as stated in Frieze and Jerrum (1997), and output the best subgraph.

2. Randomized rounding: Generate a vector

$$u \in N(0, \theta^*\bar{X} + (1-\theta^*)I)\,.$$

Then assign

$$\hat{x} = \text{sign}(u)\,.$$

Select the block $S = \{i > 0 : \hat{x}_i = \hat{x}_0\}$. Let $\tilde{S} = S$.

3. Node swapping:

- If $|\tilde{S}| > n/2$, then for each $i \in \tilde{S}$, let $\zeta(i) = \Sigma_{j\in\tilde{S}} w_{ij}$ and $\tilde{S} := \{i_1, i_2, \dots, i_{|\tilde{S}|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq \cdots \geq \zeta(i_{|\tilde{S}|})$. Then, remove node $i_{|\tilde{S}|}$ from $\tilde{S}$ and reassign $\tilde{S} := \{i_1, i_2, \dots, i_{|\tilde{S}|-1}\}$. Repeat this swapping process till $|\tilde{S}| = n/2$.
- If $|\tilde{S}| \leq n/2$, arbitrarily add $n/2 - |\tilde{S}|$ nodes from outside of $\tilde{S}$ into $\tilde{S}$. □

The following lemma is the analogue of Lemma 1.

LEMMA 4. *The total weights of edges in the subgraph induced by* $\tilde{S}$

$$
w(\tilde{S}) \geq \begin{cases} \dfrac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{|S|(|S|-1)}\, w(S) & \text{if } |S| > n/2 \\ w(S) & \text{if } |S| \leq n/2 \end{cases}. \tag{10}
$$

5. Analysis of the approximation of (DSP)

In order to analyze the quality of subgraph $\tilde{S}$, we again consider two random variables:

$$w := w(S) = \frac{1}{4} \sum_{0<i<j} w_{ij}(1 + \hat{x}_0\hat{x}_i + \hat{x}_0\hat{x}_j + \hat{x}_i\hat{x}_j)$$

$$m := |S|(n - |S|) = \frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4} = \frac{1}{4} \sum_{i>0,\, j>0} (1 - \hat{x}_i\hat{x}_j) \,.$$

The rest analysis is parallel to that of CGBP with one exception: while $\beta(\theta)$, $R(\theta)$, etc. remain the same as before, $\alpha(\theta)$ needs to be re-defined.

For any given $0 < \theta < 1$, let us generate a vector $u \in N(0, \theta\bar{X} + (1-\theta)I)$, assign $\hat{x} = \text{sign}(u)$, and select the block $S(\theta) = \{i : \hat{x}_i = \hat{x}_0\}$. Then, we have

$$\mathrm{E}[w(S(\theta))] = \frac{1}{4} \sum_{0<i<j} w_{ij}\left(1 + \frac{2}{\pi}(\arcsin(\theta\bar{X}_{0i}) + \arcsin(\theta\bar{X}_{0j}) + \arcsin(\theta\bar{X}_{ij}))\right) \tag{11}$$

Before we continue our analysis, we present some technique lemmas:

LEMMA 5. *For* $0 < \theta \leqslant 1$ *and* $0 \leqslant x \leqslant 1$, $x \arcsin(\theta) \geqslant \arcsin(\theta x)$.

Proof. It is seen that the inequality held when $x = 0$. Now we prove for $0 < x \leqslant 1$ that

$$\frac{\arcsin(\theta)}{\theta} \geqslant \frac{\arcsin(\theta x)}{\theta x} \,.$$

Here we prove that the function $g(x) = \arcsin(x)/x$ is non-decreasing in $(0, 1]$, and, therefore, the above inequality held. Take the first derivative of $g(x)$ we get

$$g'(x) = -\frac{\arcsin(x)}{x^2} + \frac{1}{x\sqrt{1-x^2}} \,.$$

Let

$$h(x) = x^2 g'(x) = -\arcsin(x) + \frac{x}{\sqrt{1-x^2}} \,.$$

Since $h(0) = 0$ and

$$h'(x) = x^2(1-x^2)^{-3/2} \geqslant 0 \,,$$

we get $h(x) \geqslant 0$ for $0 \leqslant x < 1$. Therefore, $g'(x) \geqslant 0$ and then $g(x)$ is non-decreasing in $(0, 1]$. □

LEMMA 6. *For* $0 < \theta \leqslant 1$, *the minimizer of function*

$$f(x) = \frac{1 + \frac{2}{\pi}\arcsin(\theta) + \frac{4}{\pi}\arcsin(\theta x)}{2 + 2x}$$

in $(-1, 1]$ *will be obtained at* $x \geqslant 0$.

Proof. It will be sufficient if we can prove $f(x) \leqslant f(0)$ and $f(-x) \geqslant f(0)$ for all $0 \leqslant x \leqslant 1$. Now, suppose $x > 0$. By Lemma 5, we have

$$\frac{1 + \frac{2}{\pi}\arcsin(\theta)}{2} \geqslant \frac{2}{\pi}\arcsin(\theta) \geqslant \frac{2}{\pi}\,\frac{\arcsin(\theta x)}{x}\,,$$

Then it is easy to see

$$f(x) = \frac{1 + \frac{2}{\pi}\arcsin(\theta) + \frac{4}{\pi}\arcsin(\theta x)}{2 + 2x} \leqslant \frac{1 + \frac{2}{\pi}\arcsin(\theta)}{2} = f(0)\,,$$

and

$$f(-x) = \frac{1 + \frac{2}{\pi}\arcsin(\theta) - \frac{4}{\pi}\arcsin(\theta x)}{2 - 2x} \geqslant \frac{1 + \frac{2}{\pi}\arcsin(\theta)}{2} = f(0)\,. \qquad \square$$

REMARK. Using the same idea we can prove that Lemma 6 holds for

$$f(x) = \frac{1 + \frac{6}{\pi}\arcsin(\theta x)}{1 + 3x}$$

and

$$f(x) = \frac{1 + \frac{2}{\pi}\arcsin(\theta x)}{1 + x}\,.$$

Denote by

$$a = \bar{X}_{0i}\,, \qquad b = \bar{X}_{0j}\,, \quad \text{and} \quad c = \bar{X}_{ij}\,.$$

Note that $\bar{X}_{0i}$, $\bar{X}_{0j}$ and $\bar{X}_{ij}$ are three off-diagonal components of a 3×3 positive semidefinite matrix whose diagonal components are all ones. The following lemma is a generalization of Lemma 6.3 in Goemans and Williamson (1995).

LEMMA 7. *For any* $0 < \theta \leqslant 1$,

$$1 + \frac{2}{\pi}(\arcsin(\theta a) + \arcsin(\theta b) + \arcsin(\theta c)) \geqslant \alpha(\theta)(1 + a + b + c)$$

for all

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \geqslant 0\,,$$

where

$$\alpha(\theta) = \min_{-1/3 < y \leqslant 1} \frac{1 + \frac{6}{\pi}\arcsin(\theta y)}{1 + 3y}\,. \tag{12}$$

Proof. Consider the minimization problem of

$$\begin{aligned} &\text{Minimize} \quad 1 + \frac{2}{\pi}(\arcsin(\theta a) + \arcsin(\theta b) + \arcsin(\theta c)) - \alpha(\theta)(1 + a + b + c) \\ &\text{subject to} \quad \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \succeq 0 . \end{aligned} \tag{13}$$

We want to prove that 0 is its minimal value.

The constraints of the problem can be equivalently written as

$$a, b, c \in [-1, 1] \quad \text{and} \quad a^2 + b^2 + c^2 - 2abc \leqslant 1 .$$

If one of a, b, c, say a, is -1, then we can verify that $b + c = 0$, and the objective value is

$$1 - \frac{2}{\pi}\arcsin(\theta) \geqslant 0 .$$

If one of a, b, c, say a, is 1, then we can verify that $b - c = 0$. If $b = c \geqslant 0$, by the definition of $\alpha(\theta)$, we have

$$\begin{aligned} &1 + \frac{2}{\pi}\arcsin(\theta) + \frac{4}{\pi}\arcsin(\theta b) \\ &\quad = \frac{1}{3}\cdot\left(1 + \frac{6}{\pi}\arcsin(\theta)\right) + \frac{2}{3}\cdot\left(1 + \frac{6}{\pi}\arcsin(\theta b)\right) \\ &\quad \geqslant \frac{1}{3}\cdot\alpha(\theta)\cdot(1 + 3) + \frac{2}{3}\cdot\alpha(\theta)\cdot(1 + 3b) \\ &\quad = \alpha(\theta)\cdot(2 + 2b) . \end{aligned} \tag{14}$$

By Lemma 6, we can see (14) also holds when $b < 0$. Therefore, the objective value

$$1 + \frac{2}{\pi}(\arcsin(\theta) + 2\arcsin(\theta b)) - \alpha(\theta)(2 + 2b) \geqslant 0 .$$

Now, we need only to prove that the objective value of (13) subject to the set

$$\{(a, b, c) : a, b, c \in (-1, 1) \text{ and } a^2 + b^2 + c^2 - 2abc \leqslant 1\}$$

is equal to or greater than 0.

Actually, we prove that the objective value of (13) subject to a larger set

$$\{(a, b, c) : a, b, c \in (-1, 1) \text{ and } a + b + c \geqslant -3/2\}$$

is equal to or greater than 0. Why is this set larger than the previous one? Consider another minimization problem

$$\begin{aligned} &\text{Minimize} \quad a + b + c \\ &\text{subject to} \quad \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \succeq 0 . \end{aligned} \tag{15}$$

This is a convex minimization problem and one can verify that its minimal value is $-3/2$ at $a = b = c = -1/2$. Thus,

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \geqslant 0$$

implies that

$$a + b + c \geqslant -3/2 .$$

Finally consider the third minimization problem

$$\begin{array}{ll} \text{Minimize} & 1 + \dfrac{2}{\pi}(\arcsin(\theta a) + \arcsin(\theta b) + \arcsin(\theta c)) - \alpha(\theta)(1 + a + b + c) \\ \text{subject to} & a + b + c \geqslant -3/2 \\ & -1 < a, b, c < 1 . \end{array} \tag{16}$$

One can see all its *KKT* points satisfy

$$a^2 = b^2 = c^2 .$$

If $a = b = c$, we can see the lemma is true for $a > -\frac{1}{3}$ by the definition of $\alpha(\theta)$; the lemma is true for $-\frac{1}{2} \leqslant a \leqslant -\frac{1}{3}$ since the first part of the objective function is nonnegative and the second part is nonpositive. Otherwise, suppose $b = -c$. If $a \geqslant 0$, the objective value is

$$\begin{aligned} 1 + \frac{2}{\pi}\arcsin(\theta a) - \alpha(\theta)(1 + a) &= \frac{2}{3} + \frac{1}{3} \cdot \left(1 + \frac{6}{\pi}\arcsin(\theta a) - \alpha(\theta)(1 + a)\right) \\ &\geqslant \frac{2}{3} + \frac{1}{3} \cdot \alpha(\theta) \cdot (1 + 3a) - \alpha(\theta)(1 + a) \\ &\geqslant 0 . \end{aligned}$$

This inequality also holds for $a < 0$ by the remarks we made following Lemma 6. □

Now let $\alpha(\theta)$ be as in (12), $\beta(\theta)$ be as in (4), and the random variable $z(\gamma)$ be as in (7). Let θ^* and R^* be the optimizer and optimal value, respectively, to the following:

$$\max_{0 \leqslant \theta \leqslant 1} R(\theta) := \frac{\alpha(\theta)}{\beta(\theta)^2}(1 - \sqrt{1 - \beta(\theta)})^2 .$$

Furthermore, let $\alpha := \alpha(\theta^*)$, $\beta := \beta(\theta^*)$, Then, Lemma 2 holds for (DSP) and its relaxation (DSDP); and Lemma 3 also holds for $w(\tilde{S})$, i.e.,

$$w(\tilde{S}) \geqslant \frac{n-2}{n-1} \cdot R^* \cdot w^* ,$$

where

$$R^* = \frac{\alpha}{\beta^2}(1 - \sqrt{1 - \beta})^2 .$$

The proof of the latter for case $1/2 \leqslant \delta \leqslant 1$ is identical to the proof of Lemma 3; and the proof of case $0 \leqslant \delta \leqslant 1/2$ is from (10):

$$
\begin{aligned}
w(\tilde{S}) &\geqslant w(S) \\
&\geqslant (\alpha + \gamma\beta - 4\gamma\delta(1-\delta)) \cdot w^* \\
&\geqslant \frac{\alpha + \gamma\beta - 4\gamma\delta(1-\delta)}{4(1-\delta)^2} \cdot w^* \\
&\geqslant R^* \cdot w^* .
\end{aligned}
$$

The last inequality holds by switching δ and $1-\delta$ in the proof of the first case.

Now we have the second main result:

THEOREM 2. *The worst case performance ratio of the algorithm for the DENSE-$n/2$-SUBGRAPH is at least* 0.519 *for sufficiently large* n.

Proof. For n sufficiently large, we verified that for $\theta = 0.84$, $\alpha(\theta) \geqslant 0.7079$ and $\beta(\theta) \geqslant 0.9719$, therefore,

$$
R^* \geqslant R(\theta) = \frac{\alpha(\theta)}{\beta(\theta)^2}\,(1 - \sqrt{1-\beta(\theta)})^2 \geqslant 0.5193 .
$$

Following the rest of proof of Frieze and Jerrum (1997), we have proved the theorem. □

6. Strengthened SDP relaxation of (DSP)

Our strengthened SDP relaxation of (DSP) is

$$
\text{(DSDP)} \qquad
\begin{aligned}
w_{SD} := \text{Maximize} \quad & \frac{1}{4}\sum_{0<i<j} w_{ij}(1 + X_{0i} + X_{0j} + X_{ij}) \\
\text{subject to} \quad & \sum_{j=1}^{n} X_{0j} = 0 , \\
& \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij} = 0 , \\
& X_{jj} = 1,\ j = 0, \ldots, n,\ X \succcurlyeq 0 .
\end{aligned}
\qquad (17)
$$

Here, the unknown $X \in R^{(n+1)\times(n+1)}$, row and column indexed from 0 to n. Note that this SDP relaxation is a strengthened version of the SDPs used in Srivastav and Wolf (1998), where there is no constraint $\sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij} = 0$; and used in (9), where there is no constraint $\sum_{j=1}^{n} X_{0j} = 0$. Let $\bar{X}$ be an optimal solution of the strengthened SDP relaxation.

In order to analyze the quality of bisection $\tilde{S}$ resulted from the same rounding procedure used earlier, we consider three random variables:

$$w := w(S) = \frac{1}{4}\sum_{0<i<j} w_{ij}(1 + \hat{x}_0\hat{x}_i + \hat{x}_0\hat{x}_j + \hat{x}_i\hat{x}_j)\,,$$

$$p := n - |S| = \frac{1}{2}\sum_{j>0}(1 - \hat{x}_0\hat{x}_j)\,,$$

and

$$m := |S|(n - |S|) = \frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4} = \frac{1}{4}\sum_{i>0,\,j>0}(1 - \hat{x}_i\hat{x}_j)\,.$$

Let us recall and introduce more notations. For a given $0 \leqslant \theta \leqslant 1$, let $\alpha(\theta)$ be as in (12) and $\beta(\theta)$ be as in (4), and let θ^* and R^* be the optimizer and optimal value, respectively, to the following:

$$\max_{0\leqslant\theta\leqslant1} R(\theta) := \frac{\alpha(\theta)(2\eta(\theta) - \eta(\theta)^2)}{1 + 4\eta(\theta) - \beta(\theta)(1 + 2\eta(\theta))}$$

where

$$\eta(\theta) = \frac{\sqrt{(9 - 5\beta(\theta))(1 - \beta(\theta))} - (1 - \beta(\theta))}{2(2 - \beta(\theta))}\,. \tag{18}$$

Then, let $\alpha := \alpha(\theta^*)$, $\beta := \beta(\theta^*)$ and $\eta = \eta(\theta^*)$. Note that

$$0 \leqslant \alpha \leqslant 1\,, \qquad 0 \leqslant \beta \leqslant 1\,, \quad \text{and} \quad 0 \leqslant \eta \leqslant 1/2\,.$$

LEMMA 8. *Our approximation method yields S satisfying the following three inequalities:*

$$E[w] \geqslant \alpha \cdot w_{SD} \geqslant \alpha \cdot w^*\,,$$

$$E[p] \geqslant \beta \cdot \frac{n}{2}\,,$$

$$E[m] \geqslant \beta \cdot \frac{n^2}{4}\,.$$

Proof. The first and third inequalities are proved earlier. To prove the second inequality, let

$$\alpha' = \min_{-1\leqslant y<1} \frac{2}{\pi} \cdot \frac{\arcsin(\theta^*) - \arcsin(\theta^* y)}{1 - y}$$

and noting that

$$\sum_{j>0} \bar{X}_{0j} = 0\,,$$

we derive

$$\begin{aligned}
\mathrm{E}[n - |S|] &= \frac{1}{2}\sum_{j>0}\left(1 - \frac{2}{\pi}\arcsin(\theta^*\bar{X}_{0j})\right)\\
&= \frac{1}{2}\sum_{j>0}\left(1 - \frac{2}{\pi}\arcsin(\theta^*) + \frac{2}{\pi}\arcsin(\theta^*) - \frac{2}{\pi}\arcsin(\theta^*\bar{X}_{0j})\right)\\
&\geqslant \frac{1}{2}\sum_{j>0}\left(1 - \frac{2}{\pi}\arcsin(\theta^*) + \alpha'(1 - \bar{X}_{0j})\right)\\
&= \left(1 - \frac{2}{\pi}\arcsin(\theta^*) + \alpha'\right)\cdot\frac{n}{2}\\
&\geqslant \beta\cdot\frac{n}{2}\,.
\end{aligned}$$

□

Consider a new random variable

$$z(\eta, \gamma) := \frac{w}{w^*} + 2\eta\gamma\,\frac{2p}{n} + \gamma\,\frac{4m}{n^2}\,, \tag{19}$$

where η is given by (18) and

$$\gamma = \frac{\alpha}{1 + 4\eta - \beta(1 + 2\eta)}\,. \tag{20}$$

By Lemma 8, we have

$$\mathrm{E}[z(\eta, \gamma)] \geqslant \alpha + 2\eta\gamma\beta + \gamma\beta \quad \text{and} \quad z(\eta, \gamma) \leqslant 1 + 4\eta\gamma + \gamma(1 - \eta)^2\,.$$

Now we prove the last lemma:

LEMMA 9. *If random variable $z(\eta, \gamma)$ fulfills its expectation, i.e., $z(\eta, \gamma) \geqslant \alpha + (2\eta + 1)\gamma\beta$, then*

$$w(\tilde{S}) \geqslant \frac{n-2}{n-1}\cdot R^*\cdot w^*\,,$$

where recall that

$$R^* = \frac{\alpha(2\eta - \eta^2)}{1 + 4\eta - \beta(1 + 2\eta)}\,.$$

Proof. Suppose

$$w(S) = \lambda w^* \quad \text{and} \quad |S| = \delta n\,,$$

which from (19) and $z(\eta, \gamma) \geqslant \alpha + (2\eta + 1)\gamma\beta$ implies that

$$\begin{aligned}
\lambda &\geqslant \alpha + (2\eta + 1)\gamma\beta - 4\eta\gamma(1 - \delta) - 4\gamma\delta(1 - \delta)\\
&= \alpha + (2\eta + 1)\gamma\beta - 4\eta\gamma + 4\delta\gamma(\eta - 1) + 4\delta^2\gamma\,.
\end{aligned}$$

Consider the case $1/2 \leqslant \delta \leqslant 1$. Applying Lemma 4, we see that

$$
\begin{aligned}
w(\tilde{S}) &\geqslant \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{\delta n(\delta n-1)} \cdot w(S) \\
&= \frac{n-2}{n-\frac{1}{\delta}} \cdot \frac{1}{4\delta^2} \cdot w(S) \\
&\geqslant \frac{n-2}{n-1} \cdot \frac{1}{4\delta^2} \cdot w(S) \\
&= \frac{n-2}{n-1} \cdot \frac{\lambda}{4\delta^2} \cdot w^* \\
&\geqslant \frac{n-2}{n-1} \cdot \frac{\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma + 4\delta\gamma(\eta-1) + 4\delta^2\gamma}{4\delta^2} \cdot w^* \\
&= \frac{n-2}{n-1} \cdot \left(\frac{\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma}{4\delta^2} + \frac{\gamma(\eta-1)}{\delta} + \gamma \right) \cdot w^* \\
&\geqslant \frac{n-2}{n-1} \cdot \left(\gamma - \frac{\gamma^2(1-\eta)^2}{\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma} \right) \cdot w^* .
\end{aligned}
$$

The last inequality follows from simple calculus that

$$
\delta = \frac{\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma}{2\gamma(1-\eta)} = \frac{1}{2(1-\eta)}
$$

(the second equality above is from (20) that $\gamma = \alpha + (2\eta+1)\gamma\beta - 4\eta\gamma$), yields the minimal value for

$$
\frac{\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma}{4\delta^2} + \frac{\gamma(\eta-1)}{\delta} + \gamma
$$

in the interval $(0, +\infty)$.

Consider the case $0 \leqslant \delta \leqslant 1/2$. Applying Lemma 4, we see that

$$
\begin{aligned}
w(\tilde{S}) &\geqslant w(S) \\
&\geqslant (\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma + 4\delta\gamma(\eta-1) + 4\delta^2\gamma) \cdot w^* \\
&\geqslant (\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma - \gamma(1-\eta)^2) \cdot w^* .
\end{aligned}
$$

The last inequality follows from simple calculus that

$$
\delta = \frac{1-\eta}{2}
$$

yields the minimal value for

$$
\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma + 4\delta\gamma(\eta-1) + 4\delta^2\gamma
$$

in the interval $[0, 1/2]$.

Finally, our choice for γ in (20) makes

$$\gamma - \frac{\gamma^2(1-\eta)^2}{\alpha + (2\eta+1)\gamma\beta - 4\eta\gamma} = \alpha + (2\eta+1)\gamma\beta - 4\eta\gamma - \gamma(1-\eta)^2 = R^*\,,$$

which proves the lemma. □

Note that η given by (18) maximizes

$$\frac{\alpha(2\eta - \eta^2)}{1 + 4\eta - \beta(1+2\eta)}$$

for any given $0 \leqslant \beta \leqslant 1$ and $\alpha > 0$.

Finally, we have the third main result:

THEOREM 3. *The worst case performance ratio of the approximation algorithm for the DENSE-$n/2$-SUBGRAPH is at least* 0.586 *for sufficiently large n.*

Proof. For sufficiently large n, we verified that for $\theta = 0.89$, $\alpha(\theta) \geqslant 0.7368$ and $\beta(\theta) \geqslant 0.9621$, therefore, $\eta(\theta) = 0.1737$ and

$$R^* = \frac{\alpha(2\eta - \eta^2)}{1 + 4\eta - \beta(1+2\eta)} \geqslant 0.5866\,.$$

Following the rest of proof of Frieze and Jerrum (1997), we have the theorem proved. □

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